Variational principles for discrete maps

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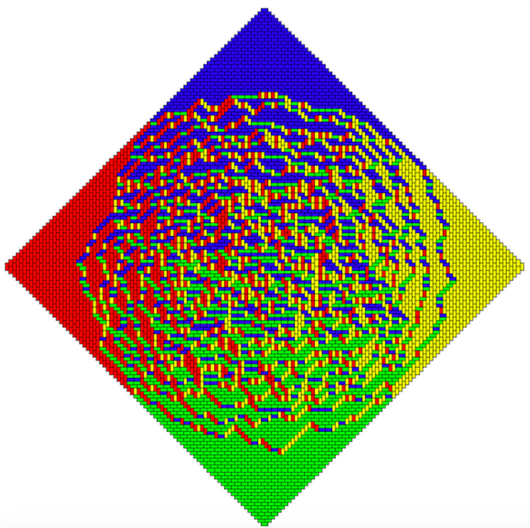
Limit shape “Aztec Diamond” for domino tilings



Colors represent the parity of the dominos

Overview

- We study random discrete maps.
- Limit shapes are a universal phenomenon.
- A variational principle explains:
  - the occurrence of limit shapes
  - the shape of the limit.

Overview

- Methods for deducing variational principles:
  - so far: based on integrability/ determinental structure of model.
  - now: based on Kirszbraun theorem and concentration inequality.
- Main result:

  Variational principle for graph homomorphisms to a tree
- First variational principle for a model of random discrete maps that is not integrable.

Object we study

Graph homomorphism $h : \mathcal{G}_1 \to \mathcal{G}_2$:

$$x \sim y \Rightarrow h(x) \sim h(y).$$

Example 1: Graph homomorphism to $\mathbb{Z}$:

$$h : \mathbb{Z}^2 \supset \Lambda \to \mathbb{Z} \qquad \text{"height function"}$$

Example 2: Graph homomorphism to 3 regular tree $\mathcal{T}$:

$$h : \mathbb{Z}^2 \supset \Lambda \to \mathcal{T} \qquad \text{"Tree-valued height function"}$$

Objects we study

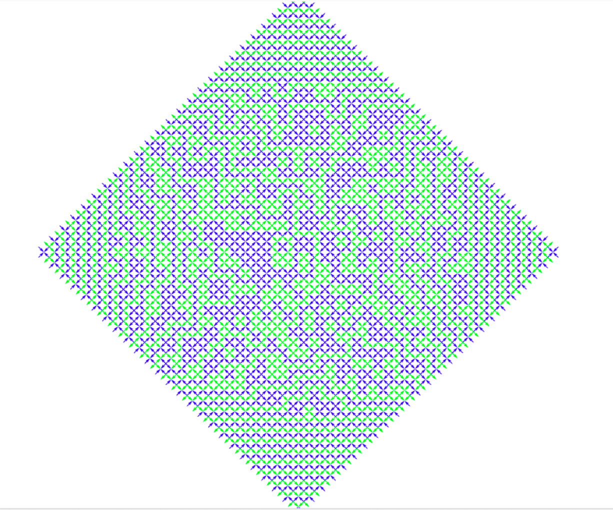
Random graph homomorphism:

1. Fix $\Lambda \subset \mathbb{Z}$
2. Fix boundary data $g : \partial\Lambda \to \mathbb{Z}$
3. Pick uniformly at random one element of

$$\left\{ h : \Lambda \to \mathbb{Z} \mid h \text{ is height function and } h_{|\partial\Lambda} = g \right\}$$

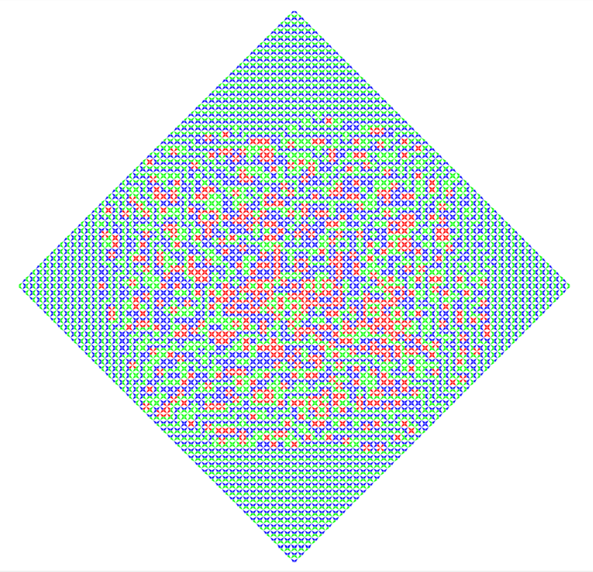
For large Λ you will see "limit shape".

Limit shape for graph homomorphism into $\mathbb{Z}$



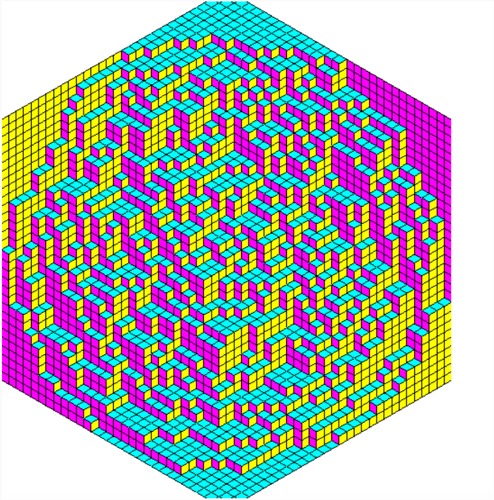
Colors represent the parity of the height in $\mathbb{Z}$

Limit shape for graph homomorphism into 3 regular tree $\mathcal{T}$



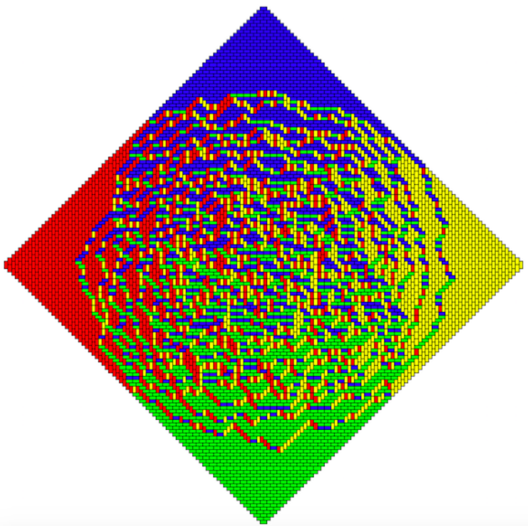
Appearance of limit shapes is a universal phenomenon

Limit shape for lozenge tilings



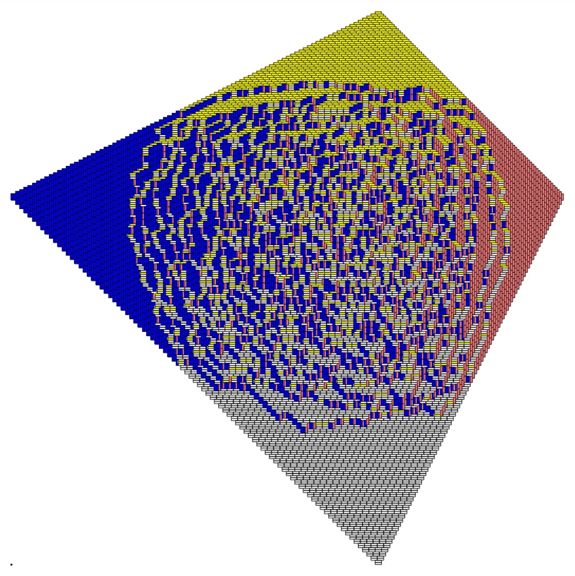
Each color represents a type of lozenge

Limit shape “Aztec Diamond” for domino tilings

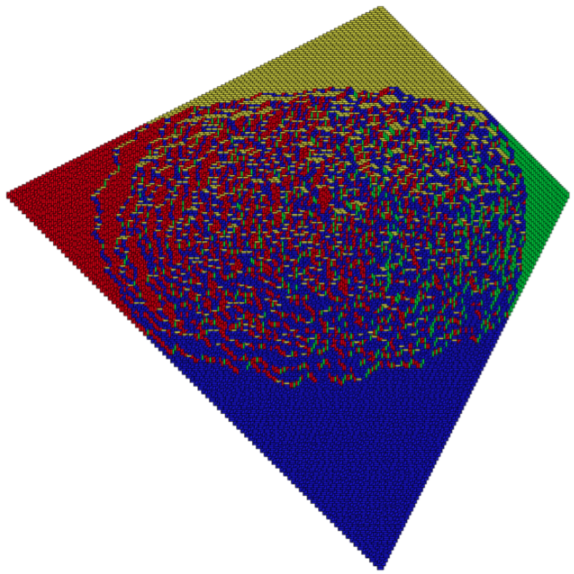


Colors represent the parity of the dominos

Limit shape for tilings by 3-1 bars



Limit shape for ribbon tilings



Variational principle for domino tilings and height functions $h : \Lambda \to \mathbb{Z}$

Preparations

Two questions:

- How many height functions $h : \Lambda \to \mathbb{Z}$ with given boundary values exist?
- What is their typical shape?

Preparations

Microscopic entropy

$$\mathrm{Ent}(\Lambda, g_{\partial\Lambda}) := \frac{1}{|\Lambda|} \ln \Big| \big\{ h : \Lambda \to \mathbb{Z} \mid h \text{ is height function and } h_{|\partial\Lambda} = g \big\} \Big|$$

Microscopic surface tension

$$\begin{aligned} \mathrm{ent}_n(s,t) := \frac{1}{n^2} \ln |\{ h : B_n \to \mathbb{Z} \mid\ & h \text{ is height function and for all} \\ & (x_1, x_2) \in \partial B_n : \quad h(x_1, x_2) \approx s \cdot x_1 + t \cdot x_2 \}| \end{aligned}$$

with $\quad B_n = \{1, \ldots, n\}^2 \quad$ and $\quad -1 \leq s, t \leq 1$

Preparations

Local surface tension

$$\mathrm{ent}(s,t) := \lim_{n\to\infty} \mathrm{ent}_n(s,t)$$

Macroscopic entropy

$$\mathrm{Ent}(R,f) := \int_R \mathrm{ent}(\nabla f(x))\ dx$$

with $\quad R \subset \mathbb{R}^2 \quad$ and $f : R \to \mathbb{R}$ 1-Lipschitz

The variational principle for domino tilings

Theorem 1 (Cohn, Kenyon, Propp '00)

Assume: $\frac{1}{n}\Lambda_n \to R$ *and* $g_{\partial\Lambda_n} \to g_{\partial R}$

Then: $\mathrm{Ent}(\Lambda_n, g_{\partial\Lambda_n}) \to \sup_{f\colon\ f_{\partial R} = g_{\partial g}} \mathrm{Ent}(R, f).$

Asymptotically characterizes the number of height-functions.

The variational principle for domino tilings

Additionally

$$\mathrm{ent}(s,t) \qquad \text{is strict concave}$$

and therefore

$$\sup_{f : f_{\partial R} = g_{\partial g}} \mathrm{Ent}(R,f) = \max_{f : f_{\partial R} = g_{\partial g}} \mathrm{Ent}(R,f) = \int_R \mathrm{ent}(\nabla f_{\mathrm{max}}(x))\, dx.$$

WHY DO LIMIT SHAPES APPEAR?

The variational principle for domino tilings

Given $f : R \to \mathbb{R}$ and $\varepsilon > 0$ define

$$B(\varepsilon, f, \Lambda_n) := \quad \{\text{height functions } h : \Lambda_n \to \mathbb{Z} \text{ that are } \varepsilon\text{-close to } f\}$$

Theorem 2 (Cohn, Kenyon, Propp '00)

$$\frac{1}{|\Lambda_n|} \ln |B(\varepsilon, f, \Lambda_n)| \to \mathrm{Ent}(R, f) + \mathrm{Error}(\varepsilon)$$

Why limit shapes appear

Combination of both theorems yields

$$\begin{aligned}\frac{1}{|\Lambda_n|}\ln|B(\varepsilon, f_{\max}, \Lambda_n)| &\approx \mathrm{Ent}(f_{\max}) \\ &\approx \mathrm{Ent}_n(\Lambda_n, g_{\partial\Lambda_n}) = \frac{1}{|\Lambda_n|}\ln|Z(\Lambda_n, g_{\partial\Lambda_n})|\end{aligned}$$

This means:
Uniform measure on $Z(\Lambda_n, g_{\partial\Lambda_n})$ concentrates around $B(\varepsilon, f_{\max}, \Lambda_n)$

Main ingredients of the proof

First ingredient: $\lim_{n\to\infty} \mathrm{ent}_n(s,t)$ exists

Second ingredient: $\lim_{n\to\infty} \mathrm{ent}_n(s,t) = \lim_{n\to\infty} \mathrm{ent}_{n,\mathtt{free}}(s,t)$

$$\mathrm{ent}_{n,\mathtt{free}}(s,t) = \frac{1}{n^2} \ln \big|\{h : B_n \to \mathbb{Z} \mid h \text{ is height function and for all } (x_1,x_2) \in \partial B_n : \ |h(x_1,x_2) - s\cdot x_1 + t\cdot x_2| \lesssim \sqrt{n}\}\big|$$

Main ingredients of the proof of Cohn, Kenyon, Propp

Cohn, Kenyon, Propp use integrability of model
Hence, $\mathrm{ent}_n(s,t)$ explicitly known.

PROOF WITHOUT USING INTEGRABILITY?

Content of recent work with Martin Tassy on ArXiv.

Derive variational principle for graph homomorphisms $h : \Lambda_n \to \mathcal{T}$.

Comparing both methods

Method based on integrability: not robust but very precise

- local surface tension $\mathrm{ent}(s, t)$ is explicitly known
- local surface tension $\mathrm{ent}(s, t)$ is strictly concave
- limit shapes f_{max} can be analyzed
- fluctuations can be analyzed (Kenyon, Okunkov, Sheffield '06)

Comparing both methods

The new method: very robust but not precise

- existence of limit $\lim_{n\to\infty} \operatorname{ent}_n(s,t)$ via compactness
- local surface tension $\operatorname{ent}(s,t)$ not explicitly known
- local surface tension $\operatorname{ent}(s,t)$ is concave, strict concavity open problem
- applies to many other models e.g. $h : \mathbb{Z}^m \to$ d-regular Tree

How is integrability substituted?

Two ingredients:

Kirszbraun theorem for graphs: when can one glue two graph homomorphisms together.

Concentration inequality

$$\mathbb{P}\left(|h(x_1, x_2) - (sx_1 + tx_2)| \geq \varepsilon\right) \lesssim \exp\left(-C\frac{\varepsilon^2}{n}\right)$$

How are those ingredients used?

Showing $\lim_{n\to\infty} \mathrm{ent}_n(t,s) = \lim_{n\to\infty} \mathrm{ent}_{n,\mathtt{free}}(t,s)$.

How is the concentration inequality deduced

For graph homomorphisms $h : B_n \to \mathbb{Z}$ several methods:

- Simple random walk on $\mathbb{Z}$ nicely behaved.......Brownian sheet.
- Sheffield: cluster swapping

For graph homomorphisms $h : B_n \to \mathcal{T}$ very difficult:

- Simple random walk on $\mathcal{T}$ not nice: not commutative and runs away

New coupling technique:

- Based on Azuma- Hoeffding inequality
- uses a very carefully adapted Glauber dynamics

Outlook

Open questions:

- Is the the local entropy is strictly concave?
- Is it possible to approximate the local entropy of a given slope in polynomial time?
- Derive a quantitative version of the variational principle

Other models:

- Use different graphs instead of $\mathcal{T}$.........connected to groups, random graphs.
- Use different graps instead of $\mathbb{Z}^2$
- Consider weighted graphs instead of $\mathcal{T}$....do homogenization.

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Thank you!