Non-conservative optimal transport

Georg Menz
UCLA, Department of Mathematics

États de la Recherche SMF
Inégalités fonctionnelles
CIRM, 20 November 2026

joint work with



Gabriela Kováčová
Reykjavík University
multi-objective optimization
mathematical finance



Niket Patel
NYU (formerly UCLA)
deep learning theory
AI for mathematics

Thank you to the organizers!

Organizing committee



Fanny Augeri
Paris Cité
large deviations
random matrices & graphs



Max Fathi
Paris Cité
Stein's method
OT & curvature



Ivan Gentil
Lyon 1
Bakry–Émery Γ-calculus
Schrödinger problem

Nathael Gozlan
Paris Cité
transport–entropy
inequalities
weak transport



Arnaud Guillin
Clermont Auvergne
Lyapunov methods, MCMC
mean-field long time

Scientific committee



Franck Barthe
Toulouse
Brascamp–Lieb
convex geometry

François Bolley
ENS Rennes
kinetic & mean-field PDE
Wasserstein contraction



Pietro Caputo
Roma Tre
mixing times, spectral gap
entropy factorization



Mylène Maïda
Lille
random matrices
Coulomb gases

Marjolaine Puel
CY Cergy
kinetic theory
diffusion limits

Thank you to all speakers and participants!





Composite of photos by ICSA (Pexels) and Patrick Kool (Unsplash)

Interacting stochastic systems









Do bees solve an optimal transport problem?

Do bees solve an optimal transport problem?

Partly. Colonies quickly shift foragers to the richest feeder; bumblebees learn near-shortest routes (traplines, a small TSP). But honey bees in a crop field did not match an ideal free distribution. *(Seeley et al. 1991; Lihoreau et al. 2012; Robinson et al. 2022)*

Do bees solve an optimal transport problem?

Partly. Colonies quickly shift foragers to the richest feeder; bumblebees learn near-shortest routes (traplines, a small TSP). But honey bees in a crop field did not match an ideal free distribution. *(Seeley et al. 1991; Lihoreau et al. 2012; Robinson et al. 2022)*

Individual optimum — equilibrium

each forager maximizes its own intake

ideal free distribution (Fretwell–Lucas 1970): nobody gains by moving ⇒ equal intake on every used patch

= Nash / Wardrop equilibrium

Social optimum — planner

the colony minimizes the *total* cost of bringing nectar home

optimal transport: one planner, the superorganism

= what a colony “should” do

Do bees solve an optimal transport problem?

Partly. Colonies quickly shift foragers to the richest feeder; bumblebees learn near-shortest routes (traplines, a small TSP). But honey bees in a crop field did not match an ideal free distribution. *(Seeley et al. 1991; Lihoreau et al. 2012; Robinson et al. 2022)*

Individual optimum — equilibrium

each forager maximizes its own intake

ideal free distribution (Fretwell–Lucas 1970): nobody gains by moving ⇒ equal intake on every used patch

= Nash / Wardrop equilibrium

Social optimum — planner

the colony minimizes the *total* cost of bringing nectar home

optimal transport: one planner, the superorganism

= what a colony “should” do

In general the two differ (price of anarchy). Is a colony a planner or a crowd?

How do they find the answer, with nobody in charge?

Largely known. Waggle-dance recruitment grows with the profitability of a source (positive feedback), foragers abandon poor sources; no single bee compares the sources. *(Seeley et al. 1991; Seeley 1995)*

Hand-over at the entrance: forager → receiver bee; if unloading is slow, the *tremble dance* recruits more receivers — load balancing. *(Seeley 1992; Seeley et al. 1996)*

How do they find the answer, with nobody in charge?

Largely known. Waggle-dance recruitment grows with the profitability of a source (positive feedback), foragers abandon poor sources; no single bee compares the sources. *(Seeley et al. 1991; Seeley 1995)*

Hand-over at the entrance: forager → receiver bee; if unloading is slow, the *tremble dance* recruits more receivers — load balancing. *(Seeley 1992; Seeley et al. 1996)*

A dynamic view: a Schrödinger bridge? (our speculation)

- reference process R: scouts searching at random; marginals: the hive and the flowers
- $\min_P H(P \mid R)$ over path measures with these marginals: the *most likely* collective traffic
- static version: entropic optimal transport (Sinkhorn)

How do they find the answer, with nobody in charge?

Largely known. Waggle-dance recruitment grows with the profitability of a source (positive feedback), foragers abandon poor sources; no single bee compares the sources. *(Seeley et al. 1991; Seeley 1995)*

Hand-over at the entrance: forager → receiver bee; if unloading is slow, the *tremble dance* recruits more receivers — load balancing. *(Seeley 1992; Seeley et al. 1996)*

A dynamic view: a Schrödinger bridge? (our speculation)

- reference process R: scouts searching at random; marginals: the hive and the flowers
- $\min_P H(P \mid R)$ over path measures with these marginals: the *most likely* collective traffic
- static version: entropic optimal transport (Sinkhorn)

Twist: foragers *gain* nectar and *burn* honey in flight $\propto$ distance $\Rightarrow$ mass changes along the paths: a **non-conservative** Schrödinger bridge — open.

The hive as a mind?

3. Is a hive a living computer? How does intelligence emerge in a superorganism?

The hive as a mind?

3. Is a hive a living computer? How does intelligence emerge in a superorganism?

   Partly. Choosing a nest site, scouts accumulate evidence by dancing, inhibit rival scouts with stop signals (cross-inhibition) and commit at a quorum — like decision circuits in the brain. *(Hölldobler & Wilson 2009; Seeley 2010; Marshall et al. 2009; Seeley et al. 2012)*

The hive as a mind?

3. Is a hive a living computer? How does intelligence emerge in a superorganism?

   Partly. Choosing a nest site, scouts accumulate evidence by dancing, inhibit rival scouts with stop signals (cross-inhibition) and commit at a quorum — like decision circuits in the brain. *(Hölldobler & Wilson 2009; Seeley 2010; Marshall et al. 2009; Seeley et al. 2012)*

4. Is the hive aware of itself? Does it have *Bewusstsein*?

The hive as a mind?

3. Is a hive a living computer? How does intelligence emerge in a superorganism?

   Partly. Choosing a nest site, scouts accumulate evidence by dancing, inhibit rival scouts with stop signals (cross-inhibition) and commit at a quorum — like decision circuits in the brain. *(Hölldobler & Wilson 2009; Seeley 2010; Marshall et al. 2009; Seeley et al. 2012)*

4. Is the hive aware of itself? Does it have *Bewusstsein*?

   Open. There is no accepted test for awareness of a colony. Individual bumblebees show emotion-like states and pain-like trade-offs; whether this amounts to sentience is still debated. *(Perry et al. 2016; Gibbons et al. 2022; Chittka 2022)*

Several hives: competition or cooperation?

Honey bees — several hives = several sources, the same flowers

Competition. Exploitative competition, strong colonies rob weak ones; drift is largely an apiary artefact. Only *passive* sharing: bees avoid flowers just emptied by others (scent marks). *(Henry & Rodet 2018; Giurfa & Núñez 1992; Stout & Goulson 2001)*

Several hives: competition or cooperation?

Honey bees — several hives = several sources, the same flowers

Competition. Exploitative competition, strong colonies rob weak ones; drift is largely an apiary artefact. Only *passive* sharing: bees avoid flowers just emptied by others (scent marks). *(Henry & Rodet 2018; Giurfa & Núñez 1992; Stout & Goulson 2001)*

Societies with several nests — one colony, many Diracs

Cooperation. Wood ants treat sister nests as food sources: food flows between nests — but locally, between neighbours, with no colony-wide planner. Argentine ants grow many trails between nests, then prune to near-Steiner networks. *(Ellis & Robinson 2016; Ellis et al. 2014; Latty et al. 2011)*

Several hives: competition or cooperation?

Honey bees — several hives = several sources, the same flowers

Competition. Exploitative competition, strong colonies rob weak ones; drift is largely an apiary artefact. Only *passive* sharing: bees avoid flowers just emptied by others (scent marks). *(Henry & Rodet 2018; Giurfa & Núñez 1992; Stout & Goulson 2001)*

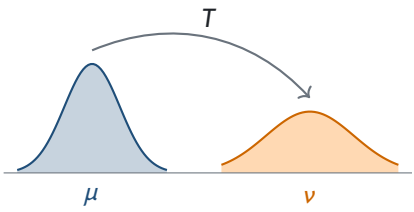
Societies with several nests — one colony, many Diracs

Cooperation. Wood ants treat sister nests as food sources: food flows between nests — but locally, between neighbours, with no colony-wide planner. Argentine ants grow many trails between nests, then prune to near-Steiner networks. *(Ellis & Robinson 2016; Ellis et al. 2014; Latty et al. 2011)*

Resource-sharing societies: a social optimum from local rules?

Then optimal transport may be the right language for resource gathering.

Classical optimal transport



Monge (1781)

$$\inf_{T_\sharp \mu = \nu} \int c\big(x, T(x)\big)\, \mu(dx)$$

Kantorovich (1942)

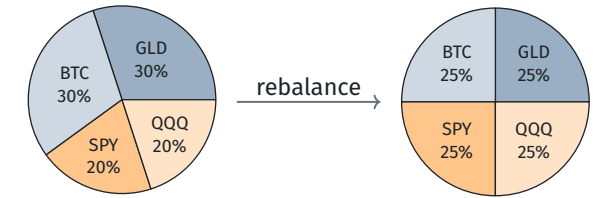
$$\inf_{\pi \in \Gamma(\mu,\nu)} \iint c(x,y)\, \pi(dx, dy)$$

$$\Gamma(\mu,\nu) = \big\{\pi : \ (\mathrm{Proj}_1)_\sharp \pi = \mu, \ (\mathrm{Proj}_2)_\sharp \pi = \nu\big\}$$

Mass is conserved: every grain of sand that leaves x arrives at y.

What if mass is lost (or gained) on the way?

A portfolio is a probability measure



current proportions μ target proportions ν

Rebalancing moves value between assets — and every trade loses value:

1 \$ of BTC buys only 0.99 \$ of GLD (bid–ask spread, fees)

Rebalancing as a transport problem

- $\pi(i,j)$ = proportion of value traded from asset i into asset j
- $m(i,j) \le 1$ = fraction of the value that survives the trade $i \to j$
- cost of the trade: $c(i,j) = 1 - m(i,j)$
- value delivered to j: $\sum_i m(i,j)\,\pi(i,j)$ — its total is not fixed in advance

$$\min_{\pi} \sum_{i,j} c(i,j)\,\pi(i,j) \quad \text{s.t.} \quad \sum_{j} \pi(i,j) = \mu_i, \qquad \frac{\sum_i m(i,j)\,\pi(i,j)}{\sum_{i,k} m(i,k)\,\pi(i,k)} = \nu_j$$

A classical coupling would fix the value *sent*, $\sum_i \pi(i,j) = \nu_j$, ignoring the loss.

$m(i,j) = q_j P_{(i,j)} / q_i$ with best-execution prices P and consistent prices q (which exist $\Leftrightarrow$ no arbitrage).

Non-conservative optimal transport

- Polish spaces $\mathcal{X}, \mathcal{Y}$, $\mu \in \mathcal{P}(\mathcal{X})$, $\nu \in \mathcal{P}(\mathcal{Y})$
- cost $c(x, y)$, mass-change factor $m(x, y) \geq 0$
- only the *shape* of the arriving mass is prescribed, not its amount
- $m \equiv 1$: classical optimal transport



Leaky Monge: $m = e^{-k\,d(x,y)}$

$$\Gamma_m(\mu,\nu) = \left\{\pi \in \mathcal{P}(\mathcal{X}\times\mathcal{Y}) : \ (\mathrm{Proj}_1)_\sharp \pi = \mu, \ \ (\mathrm{Proj}_2)_\sharp (m\pi) = \left(\int m\, d\pi\right) \nu\right\}$$

$$\inf_{\pi \in \Gamma_m(\mu,\nu)} \iint c(x,y)\, \pi(dx, dy) \qquad \text{(KP)}$$

Where mass is lost or gained on the way

Physics & logistics

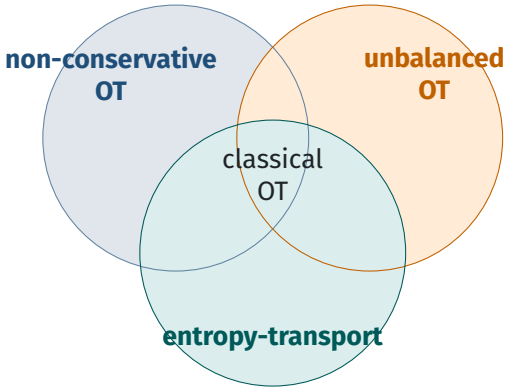
- radioactive decay, first-order reactions, drug elimination
- leaks in containers, spoilage in transport
- gain on the way: charging while travelling

  loss $\propto$ remaining mass $\rightsquigarrow$
  $m = e^{-k\,d(x,y)}$

Economics

- productivity: workers $x \longrightarrow$ jobs y, m = retention rate
- consumers and goods: m = tariffs or taxes,
  potentials φ = prices, ψ = utilities
- portfolio rebalancing: m = value kept after trading
- price of taxation, price of tariffs, price of retention:
  loss in the optimal value caused by $m < 1$

Where it sits



Unbalanced optimal transport

Transport *and* creation/destruction of mass; μ, ν may have different total mass.

Dynamic

$$\partial_t \rho + \nabla \cdot (\rho v) = \rho\, g$$

$$\min \int_0^1 \int \left(|v|^2 + \alpha\, g^2\right) d\rho_t\, dt$$

Wasserstein–Fisher–Rao = Hellinger–Kantorovich

Static: semi-couplings (π_0, π_1)

$$(\mathrm{Proj}_1)_\sharp \pi_0 = \mu, \quad (\mathrm{Proj}_2)_\sharp \pi_1 = \nu,$$

$$\min \int \bar{c}\left(x, \tfrac{d\pi_0}{d\gamma}, y, \tfrac{d\pi_1}{d\gamma}\right) d\gamma$$

$\bar{c}(x, m_0, y, m_1)$: cost of turning mass m_0 at x into m_1 at y

Link: $(\mathrm{KP}) = \inf_{Z>0} \mathrm{UOT}(\mu \to Z\nu)$ with $\bar{c} = c$ if $m_1 = m(x,y)\, m_0$, $\bar{c} = \infty$ otherwise. UOT fixes the arriving amount Z, NCOT lets it free.

Kondratyev–Monsaingeon–Vorotnikov 2016; Chizat–Peyré–Schmitzer–Vialard 2018; Liero–Mielke–Savaré 2018

Entropic (entropy-transport) optimal transport

Optimal entropy-transport (Liero–Mielke–Savaré 2018): soft marginals

$$\min_{\pi \geq 0} \int c \, d\pi \;+\; \mathcal{F}_1\big((\mathrm{Proj}_1)_\sharp \pi \,\big|\, \mu\big) \;+\; \mathcal{F}_2\big((\mathrm{Proj}_2)_\sharp \pi \,\big|\, \nu\big), \qquad \mathcal{F}(\gamma_1|\gamma_2) = \int F\Big(\tfrac{d\gamma_1}{d\gamma_2}\Big) d\gamma_2$$

- $F = \iota_{\{1\}}$ (0 at 1, ∞ else): classical optimal transport
- $F(s) = s \log s - s + 1$ and a suitable c: the Hellinger–Kantorovich distance
- NCOT fits only after replacing the second marginal by its *normalized m-weighted* version

$$T(\pi) = \frac{(\mathrm{Proj}_2)_\sharp (m\,\pi)}{\int m \, d\pi}, \qquad \min \int c \, d\pi + \mathcal{F}\big((\mathrm{Proj}_1)_\sharp \pi \,|\, \mu\big) + \mathcal{F}\big(T(\pi) \,|\, \nu\big)$$

Not entropic *regularization* $\varepsilon \int \pi \log \pi$ (Schrödinger 1931; Cuturi 2013): that comes later.

Comparison: NCOT and unbalanced optimal transport

	non-conservative OT	**unbalanced OT**
unknown	one plan π	pair (π_0, π_1) / (ρ, v, g)
mass change	prescribed: $\pi_1 = m\,\pi_0$	chosen, penalized (Fisher–Rao, entropy)
target	shape fixed, amount free	amount fixed (or soft marginals)
cost	c and m separated	$\bar{c}(x, m_0, y, m_1)$ intertwined
duality	strong; potentials $\varphi + (\psi - \int \psi\, dv)\, m$	general $\bar{c}$: subtle; entropy-transport ✓
maps	✓ in two regimes; regularity open	—
dynamics	Lagrangian ✓, Eulerian ×	Eulerian (Wasserstein–Fisher–Rao) ✓
geometry	asymmetric, no distance	Hellinger–Kantorovich is a metric
Sinkhorn	finite spaces ✓, continuum open	unbalanced Sinkhorn ✓
Schrödinger bridge	non-conservative bridge: open	with killing (Chen–Georgiou–Pavon 2022)
stability	in (μ, v, m): open	—

Existence of optimal plans

Theorem

$\mathcal{X}, \mathcal{Y}$ Polish, c lower semi-continuous and bounded below, m continuous with

$$0 < \underline{m} \leq m(x, y) \leq \overline{m} < \infty.$$

Then $\Gamma_m(\mu, \nu) \neq \varnothing$ and (KP) has a minimizer.

Proof idea: direct method. The second marginal is controlled by the first,

$$\pi_n(\mathcal{X} \times B) \;\leq\; \tfrac{1}{\underline{m}} \int_{\mathcal{X} \times B} m \, d\pi_n \;\leq\; \tfrac{\overline{m}}{\underline{m}} \, \nu(B),$$

so minimizing sequences are tight (Prokhorov), and both constraints pass to the limit because m is bounded and continuous.

Why the lower bound on m matters

Without $\underline{m} > 0$, mass can escape to infinity:

- $\mathcal{X} = \{0, 1\}$, $\mathcal{Y} = \mathbb{N}$, $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$, $\nu(\{j\}) = 2^{-j}$
- $m(0, j) = 1$, $m(1, j) = 4^{-j}$, $\quad c(0, j) = 0$, $c(1, j) = \frac{1}{j}$
- send the mass at 1 to $j = n$: almost nothing arrives, so the constraint can be repaired from 0
- cost $= \frac{1}{2n} \to 0$, so $\inf(\mathrm{KP}) = 0$
- but every admissible plan moves the mass $\frac{1}{2}$ at 1 somewhere: cost > 0

The infimum is not attained: half of the mass leaks away where m is small.

Duality

Formal Lagrangian $\rightsquigarrow$ **admissible pairs** (φ, ψ):

$$\varphi(x) + \Big(\psi(y) - \int \psi \, d\nu\Big)\, m(x,y) \;\le\; c(x,y), \qquad \text{(DP)} \quad \sup_{(\varphi,\psi)} \int \varphi \, d\mu$$

- **strong duality** $\min(\text{KP}) = \sup(\text{DP})$ on Polish spaces (Hahn–Banach)
- **dual attainment** on compact spaces, $\inf m > 0$, via generalized c-transforms

$$\varphi(x) = \inf_y \big(c(x,y) - \psi(y)\, m(x,y)\big), \qquad \psi(y) = \inf_x \frac{c(x,y) - \varphi(x)}{m(x,y)}, \qquad \int \psi \, d\nu = 0$$

- $m \equiv 1$: classical dual $\varphi(x) + \psi(y) \le c(x,y)$
- the roles of μ and ν are not symmetric

Optimal transport maps

Optimal plans live on the contact set: for $(x, y) \in \operatorname{supp} \pi^*$,

$$y \text{ minimizes } c(x, \cdot) - \psi\, m(x, \cdot), \qquad x \text{ minimizes } \frac{c(\cdot, y) - \varphi}{m(\cdot, y)}.$$

Characteristic equation (μ-a.e. x):

$$\nabla_x c(x, y) - \nabla\varphi(x) \;-\; \big(c(x, y) - \varphi(x)\big)\, \nabla_x \ln m(x, y) \;=\; 0$$

- classical OT: $\nabla_x c(x, y) = \nabla\varphi(x)$, invert in y (twist condition)
- here the *value* of φ enters, not only its gradient: no classical inversion

Two regimes with optimal maps

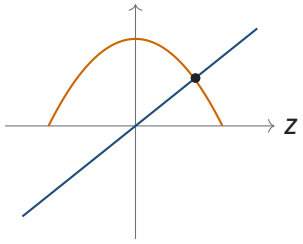
Perturbative: $c = h(|x-y|)$, $m = e^{-k\,d(|x-y|)}$, k small

$\mu \ll \mathrm{Leb} \Rightarrow$ unique map T, $\nu \ll \mathrm{Leb} \Rightarrow$ unique dual map S

Quadratic: $c = \frac{1}{2}|x-y|^2$, $m = 1 - \frac{k}{2}|x-y|^2 > 0$ the characteristic equation becomes

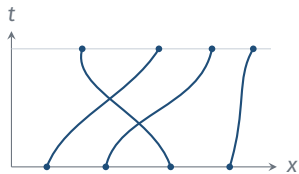
$$(x-y)\big(1-k\varphi(x)\big) = \Big(1-\tfrac{k}{2}|x-y|^2\Big)\nabla\varphi(x)$$

with $x - y = z\,\nabla\varphi(x)$: a concave parabola meets a line — exactly once.



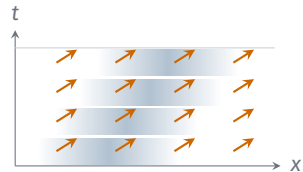
Two ways to watch moving mass

Lagrangian: follow the particles



unknown: a measure Q on *paths* γ
knows which start belongs to which end

Eulerian: stand still and watch



unknowns: density $\rho_t(z)$, velocity $v(t,z)$
how much, how fast — not which particle

Bees: radar-track every bee vs. count the bees at the hive entrance.

Classical OT: both views agree (Benamou–Brenier)

Lagrangian

$$\min\left\{ \int \int_0^1 |\dot{\gamma}_t|^2 \, dt \, Q(d\gamma) \;:\; (e_0)_\sharp Q = \mu,\ (e_1)_\sharp Q = \nu \right\} \;=\; W_2^2(\mu, \nu)$$

Eulerian (Benamou–Brenier 2000)

$$\min\left\{ \int_0^1 \int |v_t|^2 \, d\rho_t \, dt \;:\; \partial_t \rho + \nabla \cdot (\rho v) = 0,\ \rho_0 = \mu,\ \rho_1 = \nu \right\} \;=\; W_2^2(\mu, \nu)$$

- paths $\to$ fields: $\rho_t = (e_t)_\sharp Q$, $v(t,z)$ = *average* velocity of paths through z
- marginals fixed *separately*: no need to know who goes where

What is the action?

One path γ, time $0 \to 1$: $\mathcal{A}(\gamma) = \int_0^1 |\dot\gamma_t|^a \, dt$ ($a = 2$: kinetic energy; $a = 1$: length)

Least action = transport cost. For $\gamma_0 = x$, $\gamma_1 = y$, by Jensen:

$$|y - x|^a = \left| \int_0^1 \dot\gamma_t \, dt \right|^a \le \int_0^1 |\dot\gamma_t|^a \, dt, \quad \text{equality for } \gamma_t = (1-t)x + ty$$

Many particles: $\int \int_0^1 |\dot\gamma_t|^a \, dt \, Q(d\gamma)$ **Fluid:** $\int_0^1 \int |v(t,z)|^a \, \rho_t(dz) \, dt$

W_2^2 = least kinetic energy to move μ to ν in unit time.

Comparing the two actions

1. Swap the integrals (Fubini): $\int \int_0^1 |\dot\gamma_t|^a \, dt \, Q(d\gamma) = \int_0^1 \big(\int |\dot\gamma_t|^a \, Q(d\gamma) \big) dt$

2. Group the paths by where they are at time t (disintegration, $\rho_t = (e_t)_\sharp Q$):

$Q(d\gamma) = \int Q^{t,z}(d\gamma) \, \rho_t(dz)$, $\quad Q^{t,z}$ = the paths at z at time t

3. Compare pointwise in (t, z), both integrated against $\rho_t(dz)\, dt$:

$$\underbrace{\left| \int \dot\gamma_t \, dQ^{t,z} \right|^a}_{\text{Eulerian: } |v(t,z)|^a} \;\le\; \underbrace{\int |\dot\gamma_t|^a \, dQ^{t,z}}_{\text{Lagrangian: average of } |\text{speed}|^a} \qquad \text{(Jensen)}$$

Crossing ($z = 0$, $t = \frac{1}{2}$, velocities ± 2): Eulerian $0 \;\le$ Lagrangian 4

The gap is the spread of the velocities at z: zero iff the paths do not cross.

NCOT: the Lagrangian formulation

Measures Q on paths with (m is evaluated at *both ends of the same path*)

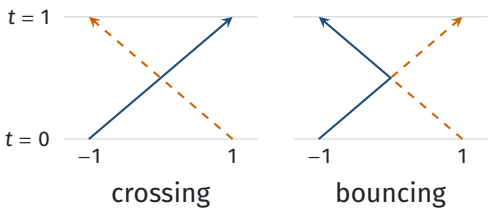
$$(e_0)_\sharp Q = \mu, \qquad (e_1)_\sharp\big(m(\gamma_0,\gamma_1)\,Q\big) = q_Q\,\nu, \qquad q_Q = \int m(\gamma_0,\gamma_1)\,Q(d\gamma)$$

Theorem (dynamic = static), $c = |y-x|^a$, $a \geq 1$

$$\inf_Q \int\!\!\int_0^1 |\dot\gamma_t|^a\,dt\,Q(d\gamma) \;=\; \inf_{\pi\in\Gamma_m(\mu,\nu)} \iint |y-x|^a\,\pi(dx,dy)$$

- $\geq$: Jensen along each path, $|\gamma_1-\gamma_0|^a \leq \int_0^1 |\dot\gamma_t|^a\,dt$
- $\leq$: move along straight lines $t \mapsto (1-t)x + ty$
- optimal map $T \Rightarrow$ optimal dynamical plan along $(1-t)x + tT(x)$

The Eulerian view forgets the pairing



- $\mu = \nu = \frac{1}{2}(\delta_{-1} + \delta_1)$, $m(x, y) = M(|y - x|)$
- same density ρ_t, same velocity field v at every time
- crossing: every path travels 2, factor $M(2)$
  bouncing: every path returns, factor $M(0)$
- $q_{Q_1} = M(2) \neq M(0) = q_{Q_2}$

The arriving mass cannot be computed from (ρ, v).

$\Rightarrow$ no Eulerian (Benamou–Brenier) statement for NCOT in general.

When paths do not cross: two Eulerian equations

$m(x,y) = M(|y-x|)$, optimal map T, $X_t(x) = (1-t)x + tT(x)$ a diffeomorphism, $v(t, X_t(x)) = T(x) - x$

- **the velocity knows both ends:** $x = z - t\,v(t,z), \quad T(x) = z + (1-t)\,v(t,z)$
- **where** the mass is: $p_t = (X_t)_\sharp \mu$,

$$\partial_t p + \nabla \cdot (p\,v) = 0$$

- **how much** mass there is: $\varrho = M(t|v|)\,p$,

$$\partial_t \varrho + \nabla \cdot (\varrho\,v) = \varrho\,|v|\,\frac{M'(t|v|)}{M(t|v|)} \qquad \text{(mass change on the way)}$$

- endpoints: $\varrho_0 = \mu$, $\varrho_1 = q\,\nu$ — the NCOT constraints reappear

In unbalanced OT the source term is a free control; here it is dictated by m.

Entropic regularization and a Sinkhorn-type algorithm

Finite spaces, $m_{ij} > 0$: $\min_{\pi \in \Gamma_m(\mu,\nu)} \sum_{ij} c_{ij}\pi_{ij} + \varepsilon \sum_{ij} \pi_{ij} \log \pi_{ij}$

- unique minimizer $\pi^\varepsilon_{ij} = \exp\big((\varphi_i + m_{ij}\,\psi_j - c_{ij})/\varepsilon\big)$
- alternate two entropic projections:
  - rows: $\pi_{ij} \mapsto \pi_{ij}\,\mu_i / \sum_l \pi_{il}$
  - columns: $\pi_{ij} \mapsto \pi_{ij}\, b_j^{m_{ij}}$, with $\sum_i m_{ij}\pi_{ij} b_j^{m_{ij}} = q\,\nu_j$, $\sum_j \nu_j \log b_j = 0$
- converges to π^ε, and $0 \le \langle c, \pi^\varepsilon\rangle - \min(\mathrm{KP}) \le \varepsilon \log(NM)$

Continuum: potentials $\varphi(x) + m(x,y)\,\psi(y)$ are not separable, and the total mass q is an extra unknown: a Schrödinger system of a new type.

Open questions: theory

- **Structure and regularity of maps:** classical $T = \nabla u$, u convex (Brenier), u solves Monge–Ampère (Caffarelli, Ma–Trudinger–Wang). Quadratic NCOT: $T(x) = x - z(x)\nabla\varphi(x)$ — a gradient? convex? a Monge–Ampère-type equation?
- **Maps beyond radial m:** is a smallness condition on $\nabla \ln m$ enough?
- **Stability:** how do optimal plans, potentials and maps depend on (μ, ν, m)? Quantitative bounds? (cf. stability of Brenier potentials, of entropic projections)
- **Non-compact spaces:** dual attainment under moment and growth conditions
- **Geometry:** is there a non-conservative Wasserstein-type geometry, and transport–entropy or functional inequalities for it?

Open questions: entropic, dynamic, bees

- **Entropic NCOT in the continuum:** existence of potentials $\varphi(x) + m(x,y)\psi(y)$, convergence and rates of the scaling iteration (semiconcavity / stability of entropic projections?)
- **Non-conservative Schrödinger problem:** $\min H(P \mid R)$ over path measures with $(e_0)_\sharp P = \mu$, $(e_1)_\sharp\big(m(X_0,X_1)P\big) = q\,\nu$ — existence, small-noise limit $\to$ NCOT?
- **Balancing problem** (bees): collecting and processing must keep pace — transport with a processing capacity at the target, balanced by local queue signals (waggle / tremble dance). Optimal? Equilibrium or social optimum?

Summary

- **Mass can change on the way:**
  prescribe the *shape* of the arriving mass, not its amount.
- **What survives:**
  existence, duality, maps in two regimes; Lagrangian ✓, Eulerian ×.
- **What is open:**
  entropic and Schrödinger versions, stability, regularity, the bees' balancing problem.



For the next generation: plenty of open problems, and nectar for everyone.


Benjamin Arras
Yupeng Bai
Aziz Ben Nejma
Pierre Bizeul
François Bolley
Charles Bordenave
Yassine Boukhateb
Thi Thu Ha Bui
Patrick Cattiaux
Louis-Pierre Chaintron
Samuel Chan-Ashing
Giovanni Conforti
Adrien Cotil
Fanch Coudreuse
Benoit Dagallier
Sandrine Dallaporta
Oscar De Wit
Fanny Delebecque
Vincent Divol
Jean Dolbeault
Katharina Eichinger
Max Fathi
Raphael Gastaldello
Ivan Gentil
Marta Gentiloni Silveri
Quentin Giton
Nathael Gozlan
Jules Grass
Arnaud Guillin
Hélène Halconruy
Ronan Herry
Baptiste Huguet
Andrea Vanessa Hurtado Quiceno
Cyril Imbert
Paul Invernizzi
Nikolai Jenki
Pierre Le Bris
Thibaut Le Gouic
Adrienne Le Meur
Joseph Lehec
Cyril Letrouit
Galyna Livshyts
Amélie Loher
Pablo López Rivera
Hugo Malamut
Andreas Malliaris
Aymeric Martin
Matteo Miannay
Manon Michel
Dionysis Milesis
Pierre Monmarché
Assylbek Olzhabayev
Francesco Pagliarin
Fabien Panloup
Francesco Pedrotti
Alberto Peiretti
Matteo Picco
Christophe Poquet
Marjolaine Puel
Eli Putterman
Ons Rameh
Julien Reygner
Shay Sadovsky
Paul-Marie Samson
Marielle Simon
Emmanouil Stavrakakis
Austin Stromme
Alexandre Surin
Maud Szusterman
Songbo Wang
Jonte Weixler
Pierre Youssef

Thank you!

References

- G. Kováčová, G. Menz, N. Patel. *Non-conservative optimal transport.* Submitted, 2026.
- F. Santambrogio. *Optimal transport for applied mathematicians.* Birkhäuser, 2015.
- L. Chizat, G. Peyré, B. Schmitzer, F.-X. Vialard. *Unbalanced optimal transport: dynamic and Kantorovich formulations.* J. Funct. Anal., 2018.
- M. Liero, A. Mielke, G. Savaré. *Optimal entropy-transport problems and a new Hellinger–Kantorovich distance.* Invent. Math., 2018.
- M. Cuturi. *Sinkhorn distances: lightspeed computation of optimal transport.* NeurIPS, 2013.
- M. Nutz. *Introduction to entropic optimal transport.* Lecture notes, 2022.

Bee references

A. Blanchet, G. Carlier, Optimal transport and Cournot–Nash equilibria, *Math. Oper. Res.* 41 (2016) 125–145. doi:10.1287/moor.2015.0719

L. Chittka, *The Mind of a Bee*, Princeton University Press, 2022.

S. D. Fretwell, H. L. Lucas, On territorial behavior and other factors influencing habitat distribution in birds, *Acta Biotheor.* 19 (1970) 16–36.

M. Gibbons, E. Versace, A. Crump, B. Baran, L. Chittka, Motivational trade-offs and modulation of nociception in bumblebees, *PNAS* 119 (2022) e2205821119.

M. Henry, G. Rodet, Controlling the impact of the managed honeybee on wild bees in protected areas, *Sci. Rep.* 8 (2018) 9308. doi:10.1038/s41598-018-27591-y

B. Hölldobler, E. O. Wilson, *The Superorganism*, W. W. Norton, 2009.

M. Lihoreau, N. E. Raine, A. M. Reynolds, R. J. Stelzer, K. S. Lim, A. D. Smith, J. L. Osborne, L. Chittka, Radar tracking and motion-sensitive cameras on flowers reveal the development of pollinator multi-destination routes over large spatial scales, *PLoS Biol.* 10 (2012) e1001392. doi:10.1371/journal.pbio.1001392

J. A. R. Marshall, R. Bogacz, A. Dornhaus, R. Planqué, T. Kovacs, N. R. Franks, On optimal decision-making in brains and social insect colonies, *J. R. Soc. Interface* 6 (2009) 1065–1074.

C. Perry (Solvi), L. Baciadonna, L. Chittka, Unexpected rewards induce dopamine-dependent positive emotion-like state changes in bumblebees, *Science* 353 (2016) 1529–1531.

S. V. J. Robinson, S. E. Hoover, S. F. Pernal, R. V. Cartar, Optimal distributions of central-place foragers: honey bee foraging in a mass flowering crop, *Behav. Ecol.* 33 (2022) 386–397. doi:10.1093/beheco/arab143

T. D. Seeley, *The Wisdom of the Hive*, Harvard University Press, 1995. T. D. Seeley, *Honeybee Democracy*, Princeton University Press, 2010.

T. D. Seeley, S. Camazine, J. Sneyd, Collective decision-making in honey bees: how colonies choose among nectar sources, *Behav. Ecol. Sociobiol.* 28 (1991) 277–290.

T. D. Seeley, P. K. Visscher, T. Schlegel, P. M. Hogan, N. R. Franks, J. A. R. Marshall, Stop signals provide cross inhibition in collective decision-making by honeybee swarms, *Science* 335 (2012) 108–111. doi:10.1126/science.1210361

Bee references (cont.)

S. Ellis, D. W. Franks, E. J. H. Robinson, Resource redistribution in polydomous ant nest networks: local or global?, *Behav. Ecol.* 25 (2014) 1183–1191. doi:10.1093/beheco/aru108

S. Ellis, E. J. H. Robinson, Internest food sharing within wood ant colonies: resource redistribution behavior in a complex system, *Behav. Ecol.* 27 (2016) 660–668. doi:10.1093/beheco/arv205

M. Giurfa, J. A. Núñez, Honeybees mark with scent and reject recently visited flowers, *Oecologia* 89 (1992) 113–117. doi:10.1007/BF00319022

T. Latty, K. Ramsch, K. Ito, T. Nakagaki, D. J. T. Sumpter, M. Middendorf, M. Beekman, Structure and formation of ant transportation networks, *J. R. Soc. Interface* 8 (2011) 1298–1306. doi:10.1098/rsif.2010.0612

T. D. Seeley, The tremble dance of the honey bee: message and meanings, *Behav. Ecol. Sociobiol.* 31 (1992) 375–383. doi:10.1007/BF00170604

T. D. Seeley, S. Kühnholz, A. Weidenmüller, The honey bee's tremble dance stimulates additional bees to function as nectar receivers, *Behav. Ecol. Sociobiol.* 39 (1996) 419–427. doi:10.1007/s002650050309

J. C. Stout, D. Goulson, The use of conspecific and interspecific scent marks by foraging bumblebees and honeybees, *Anim. Behav.* 62 (2001) 183–189. doi:10.1006/anbe.2001.1729